(4) Limits

Contents

Limits of Functions [4.1]

Intuitively limits of functions are the expected value of a function at points that can't be solved because they are undefined, e.g.

$\frac{(x-2)(x+2)}{(x-2)}$ would be undefined at x=2, however as x is made sufficiently close to 2, that value will become arbitrarily close to 4.

The Limit Generally

From early calculus the limit of $f(x)$, as x approaches a was said to be some value L, denoted $\lim_{x\to a}\left(f\left(x\right)\right)=L$

$$\forall \varepsilon > 0, \ , \exists \delta : \quad 0 < \left|x-a\right| < \delta \implies \left|f\left(x\right)-L\right| < \varepsilon \tag{1}$$

Remarks on this Definition Observe that the following statements are equivalent:

1. $x \neq c \ \wedge \ |x-a| < \delta$
2. $0 < |x-a| < \delta$
3. $|x-a| \in (0, \delta)$

Notation

If L is a limit of f at c, then it is said that:

1. f converges to L at c
2. $f(x)$ approaches L as x approaches c

   This is sometimes expressed with the symbolism $f(x) \to L$ as $x \to c$

And the following notation is used

1. $\lim_{x \to c} (f(x)) = L$
2. $\lim_{x \to c} f$

The Limit Using Cluster Points

In analysis we more or less use the same definition but we introduce the concept of cluster points to make it more rigorous.

Neighborhoods [2.2.7] A neighborhood is an interval about a value, e.g. the ε-neighborhood of a is some set $V_\varepsilon(a)$:

$$\begin{aligned} V_\varepsilon(a) = (\varepsilon - a, \varepsilon + a) &= \{x : \varepsilon - a < x < \varepsilon + a\} && (2) \\ &= \{x : -\varepsilon < x - a < \varepsilon\} && (3) \\ &= \{x : |x-a| < \varepsilon\} && (4) \end{aligned}$$

Cluster Points Let c be a real number and let A be a subset of the real numbers, c may or may not be contained by A it doesn't matter.
Take some interval around c, or rather consider the ε-neighborhood of c,
if, some value (other than c) can be found inside that interval/neighborhood that is also inside A, regardless of how small that interval is made, Then c is said to be a cluster point of A.

i.e., if the following is true
$\forall \varepsilon > 0, \ \exists x \neq c \in A \cap V_\varepsilon(c)$

then c is said to be a cluster point of A.

It basically means that there are infinitely infinitesimal points between any point in A and the value c.

Example

- The point 4 of the set $\{3, 4, 5\}$ is not a cluster point of that set because a 0.1-neighbourhood of 4 would be the set $V_{0.1}(4) = \{4\}$, this set does not contain a value $x \neq 4$ that is also inside the original set.

- The point 6 of $(1, 6) = \{x : 1 < x < 6\}$ is a cluster point of $(1, 6)$ because no matter how small a neighborhood is made around 6, there will always be values $x \neq 6$ inside that interval that are also inside $(1, 6)$

  also observe that in this case $6 \notin (1, 6)$

Definition of the Limit [4.1.4] So this is the definition that we moreso use in this unit and the one to memorise (or the Neighborhoods one seems simpler to memorise).

Let $A \subseteq \mathbb{R}$ and let c be a cluster point of A.

Now take some function:

$$f : A \to \mathbb{R} \tag{5}$$

It is said that L is a limit of f at c if:

$$\begin{aligned} &\forall \varepsilon > 0,\ \exists \delta > 0 : \\ &\qquad (x \in A \ \wedge \ 0 < |x - c| < \delta) \implies |f(x) - L| < \varepsilon \end{aligned} \tag{4.1.4}$$

What's the Distinction This is more or less the same as the typical definition given in early calculus (1), the distinction here is that we have specified that c must be a cluster point of A, this is more rigorous because c is always such that there are infinitely many values in any infinitesimal distance between intself and any $x \in A$,
So the limit will always mean a continuous approach as we expect, this is just a more thorough definition.

Definition using Neigborhoods [4.1.6] A value L is said to be the limit of f as $x \to c$, denoted $\lim_{x \to c} (f(x))$ if and only if:

For any given ε-neighbourhood of L, $V_\varepsilon(L)$
There exists a δ-neighbourhood of c, $V_\delta(L)$

such that:

If $x \neq c$ is in both A and $V_\delta(c)$
Then $f(x)$ must be within the neighbourhood $V_\varepsilon(L)$

Formally

$$\forall \varepsilon > 0, \ \exists \delta > 0 : \\ x \neq c, \ x \in A \cap V_\varepsilon(L) \implies f(x) \in V_\delta(c) \tag{4.1.6}$$

Defintions (4.1.6) and (4.1.4) are equivalent, and are both consistent with the initial less rigorous definition (1).

Only one Limit Value [4.1.5] If $f : A \to \mathbb{R}$ and c is a cluster point of A, then there is only one value L: $\lim_{x \to c} (f(x)) = L$

Using Sequences to Define Limits [4.1.8]

Now that limits are defined we can use sequences to define them as well, this will give us more tools to use later and allows a connection to be made between material of Chapter 3 and 4.

Definition A value L is said to be the limit of f as $x \to c$, denoted $\lim_{x \to c} (f(x))$ if and only if:

For every sequence (x_n) in A,

if (x_n) converges to c such that $x_n \neq c$,

Then $(f(x_n))$ converges to L

So basically, again, if x gets close to c, $f(x)$ gets close to L, but we took x from a sequence.

Divergence Criteria [4.1.9]

Now we can use the *Divergence Criteria* from [3.4.5] to determine whether or not a limit exists generally or at a point.

(a) Limit is not a Specific Value If $L \in \mathbb{R}$, then f does not have a limit at c, if and only if:

There is a sequence (x_n) in A with $x_n \neq c$, such that:

> (x_n) converges to c but the sequence $f(x_n)$ does not converge to L

(b) No Limit whatsover If $L \in \mathbb{R}$, then f does not have a limit at c, if and only if:

There is a sequence (x_n) in A with $x_n \neq c$, such that:

> (x_n) converges to c but the sequence $f(x_n)$ does not converge in $\mathbb{R}$

The Signum Function The Signum function returns the sign of the input value:

$$sgn(x) := \begin{cases} +\ 1 & \text{for x} > 0 \\ \quad 0 & \text{for x} = 0 \\ -\ 1 & \text{for x} < 0 \end{cases} \tag{4.1.10}$$
$$= \frac{x}{|x|}$$

Limit Theorems [4.2]

These are useful for calculating limits of functions, they are mostly extensions of [3.2].

Bounded Functions

Definition Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ and let $c \in \mathbb{R}$ be a cluster point of A.
It is said that f *is bounded on a neighbourhood of* c if:

> there exists a δ-neighborhood $V_\delta(c)$ and some constant value $M > 0$ such that:
>
> > $|f(x)| \leq M$ for every $x \in A \cap V_\delta(c)$

So basically a function is said to be *bounded on a neighbourhood of c* if:
for some interval (It doesn't matter how small) around c,
$f(x)$ can be contained in some interval

$$\exists\delta > 0,\ \exists M > 0 :$$
$$x \in V_\delta(c) \implies |f(x)| < M$$

So for example:

- $f(x) = x^3$ is *bounded on every neighborhood of every $x \in \mathbb{R}$* whereas,
- $g(x) = {}^1/_x$ is ***not** bounded on a neighborhood of 0* because $g(x)$ tends to infinity as $x \to 0$,
  - furthermore $g(x)$ is *bounded on **some but notall** neighborhoods of 1*, because an interval around 1 must not be drawn large enough to encapsulate 0.

Limits imply Bounded Neighbourhoods [4.2.2] A function is bounded on a neighborhood of a point that is a limit of that function.

If a function has a limit at c, then f must be *bounded on some neighborhood of c*,
this flows from the initial definitions because we know that c is a cluster point and that $(f(x))$ moves closer to L,
hence it must be possible to draw a small enough interval (e.g. horizontal lines on the y-axis) to contain all $f(x)$ defined by

Functions and Arithmetic [4.2.3]

Just like with sequences we can define arithmetic operations that relate to addition and multiplication with functions in order to manipulate them:

Let $A \subseteq \mathbb{R}$,

$$f : A \to \mathbb{R} \qquad g : A \to \mathbb{R} \qquad h : A \to \mathbb{R},\ h(x) \neq 0,\ \forall x \in A \tag{6}$$

We define the following Operations [4.2, p. 111]:

$$(f+g)(x) := f(x) + g(x) \tag{7}$$
$$(f-g)(x) := f(x) + g(x) \tag{8}$$
$$(fg)(x) := f(x) \times g(x) \tag{9}$$
$$(bf)(x) := b \times f(x) \tag{10}$$
$$\left(\frac{f}{h}\right)(x) := \frac{f(x)}{h(x)} \tag{11}$$

Limits of Function Operations [4.2.4] Because the limit of a function is essentially the expected value of the function around that value, it stands to reason that the limit will distribute over the basic operations:

Let the functions be defined as they were in (6) and let $c \in \mathbb{R}$ be a custer point of A.

$$\lim_{x \to c}(f) = L \qquad \lim_{x \to c}(g) = M \quad \lim_{x \to c}(h) = H \neq 0 \tag{12}$$

Then the limits are:

$$\lim_{x \to c}(f+g) = \lim_{x \to c}(f) + \lim_{x \to c}(g) = L + M \tag{13}$$
$$\lim_{x \to c}(f-g) = \lim_{x \to c}(f) - \lim_{x \to c}(g) \quad = x - y \tag{14}$$
$$\lim_{x \to c}(c \cdot f) = c \cdot \lim_{x \to c}(f) \qquad = c \cdot x \tag{15}$$
$$\lim_{x \to c}(f \times g) = \lim_{x \to c}(f) \times \lim_{x \to c}(g) \quad = x \times y \tag{16}$$
$$\lim_{x \to c}(f/h) = \lim_{x \to c}(f) \div \lim_{x \to c}(h) \quad = {}^{x}/_{y} \tag{17}$$

Limit Theorems

The rest of the chapter just provides values of varios limits.

Let the functions be defined as they were in (6) and let $c \in \mathbb{R}$ be a custer point of A.

Limits Captured in Intervals [4.2.6]

if $f(x) \in [a, b]$ for all $x \in A$, $x \neq c$, and $\lim_{x \to c}(f)$ exists,

then $f(x) \in [a, b]$

Squeeze Theorem [4.2.7] if [4.2.6] is extended to functions, then we have the squeeze theorem:

> *if* g is within an interval defined by the functions f and h:

$$f(x) \leq g(x) \leq h(x), \quad \forall x \in A, \ x \neq c \tag{18}$$

> *then* the limit of g must also be 0

$$\lim_{x \to c} (g) = L \tag{19}$$

A Positive Limit implies a neighbourhood with Positive Values Let $A \subseteq \mathbb{R}$ and let $c \in \mathbb{R}$ be a cluster point of A as in (3.4.6) above.

> *If*:

$$\lim_{x \to c} (f) > 0 \tag{20}$$

> *Then:*
>
> there is a neighborhood $V_\delta(c)$ such that $f(x) > 0, \ \forall x \in A \cap V_\delta(c)$

This also holds for negative values and basically all it says, in more rigorous language, is that if the limit point is above the x-axis then there's gotta be points to the left and right that are above the x-axis as well (because the whole cluster point thing means everything can be arbitrarily small).

Although this may start to seem a little pointless, the idea of making the definitions this rigorous is like writing code in a scripting language, by using this very precise language, the logical consequences give us exactly the concept that we want, even though we need to take a longer or alternate path to get to that concept than we would otherwise would generally take in order to describe the concept.

Extensions of the Limit Concept [4.3]

These are written in a particularly convoluted fashion, however if the preceeding material is understood the textbook can be used more or less as a reference, hence these notes will be brief.

One-Sided Limits [4.3.1]

Definition [4.3.1] Let $c \in \mathbb{R}$ be a cluster point of $A \cap (c, \infty) = \{x \in A \ : \ x > c\}$
It is said that L is a *Right-hand limit of f at c* and it is written:

$$\lim_{x \to c^+} (f) = L \tag{4.3.1}$$

This can be extended to left-hand limits as well.

Definition in Term of Sequences [4.3.2] As above it is said that L is a *Right-hand limit of f at c* if:

Every sequence (x_n) in A that converges to c is such that $f(x_n)$ converges to L, given that $x_n > c, \ \forall n \in \mathbb{N}$

Limit must be equal on both sides A limit is defined only if the limit is equal from both directions

$$\lim_{x \to c} (f) = L \iff \lim_{x \to c^+} (f) = L = \lim_{x \to c} (f) \tag{3.4.3}$$

Infinite Limits [4.3.5]

Let $c \in \mathbb{R}$ be a cluster point of A,
It is aid that f tends to ∞ as $x \to c$, and it is written:

$$\lim_{x \to c} (f) = \infty \tag{4.3.5}$$

If $\forall \alpha \in \mathbb{R}, \ \ \exists \delta > 0$:
$$0 < |x - c| < \delta \implies f(x) > \alpha, \quad \forall x \in A$$

One-Sided Limits to Infinity [4.3.8] Let $c \in \mathbb{R}$ be a cluster point of $A \cap (c, \infty) = \{x \in A \ : \ x > c\}$,
It is aid that f tends to ∞ as $x \to c^+$, and it is written:

$$\lim_{x \to c} (f) = \infty \tag{4.3.8}$$

If $\forall \alpha \in \mathbb{R}, \ \ \exists \delta > 0$:
$$0 < x - c < \delta \implies f(x) > \alpha, \quad \forall x \in A$$

Ordered Functions If $f(x) < g(x)$, then:

$$\lim_{x \to c} (f) = \infty \implies \lim_{x \to c} (g) = \infty \tag{4.3.7 (a)}$$
$$\lim_{x \to c} (g) = -\infty \implies \lim_{x \to c} (f) = -\infty \tag{4.3.7 (b)}$$

Limits at Infinity [4.3.10]

It is also useful to talk about limits as x tends to ∞

Let $(a, \infty) \subseteq A \subseteq \mathbb{R}$ for some $ain\mathbb{R}$

It is aid that the limit of f as $x \to \infty$ is L, and it is written:

$$\lim_{x \to \infty} (f) = L \tag{4.3.10}$$

If $\forall \varepsilon > 0,\ \exists K > 0$:

$x > K \implies |f(x) - L| < \varepsilon$

Limits at Infinity in Terms of Sequences [4.3.11] equivalently to (4.3.10), the definition can be expressed in terms of sequences:

> Every sequence (x_n) in $A \cap (a, \infty)$ that has $\lim(x_n) = \infty$ is such that the sequence $(f(x_n))$ converges to L

Infinite Limits at Infinity So this basically combines [4.3.10] with [4.3.5]

Let $(a, \infty) \subseteq A \subseteq \mathbb{R}$ for some $a \in \mathbb{R}$

It is aid that f tends to ∞ as $x \to \infty$, and it is written:

$$\lim_{x \to \infty} (f) = \infty \tag{4.3.13}$$

If $\forall \varepsilon > 0,\ \exists K > \alpha$:

$x > K \implies f(x) > \alpha$

Infinite Limits at Infinity in Terms of Sequences [4.3.14] equivalently to (4.3.13), the definition can be expressed in terms of sequences:

> Every sequence (x_n) in $A \cap (a, \infty)$ that has $\lim(x_n) = \infty$ is such that the limit of the sequence of function values $\lim(f(x_n)) = \infty$

Ratios of Functions This result uses (4.3.14) to restate (3.6.5) in terms of functions:

If $g(x) > 0\ \forall x > a$ and $L \neq 0$ is defined:

$$lim_{x \to \infty} \left(\frac{f(x)}{g(x)} \right) \tag{4.3.15}$$

then,

$$L > 0 \implies \lim_{x \to \infty} (f) = \infty \iff \lim_{x \to \infty (g) = \infty} \tag{4.3.15 (i)}$$

$$L < 0 \implies \lim_{x \to \infty} (f) = -\infty \iff \lim_{x \to \infty (g) = \infty} \tag{4.3.15 (ii)}$$