Any function which is analytic at a point z_0 must have a Taylor series about z_0. For, if f is analytic at z_0, it is analytic throughout some neighborhood $|z - z_0| < \varepsilon$ of that point (Sec. 24); and ε may serve as the value of R_0 in the statement of Taylor's theorem. Also, if f is entire, R_0 can be chosen arbitrarily large; and the condition of validity becomes $|z - z_0| < \infty$. The series then converges to $f(z)$ at each point z in the finite plane.

When it is known that f is analytic everywhere inside a circle centered at z_0, convergence of its Taylor series about z_0 to $f(z)$ for each point z within that circle is ensured; no test for the convergence of the series is even required. In fact, according to Taylor's theorem, the series converges to $f(z)$ within the circle about z_0 whose radius is the distance from z_0 to the nearest point z_1 at which f *fails* to be analytic. In Sec. 65, we shall find that this is actually the largest circle centered at z_0 such that the series converges to $f(z)$ for all z interior to it.

In the following section, we shall first prove Taylor's theorem when $z_0 = 0$, in which case f is assumed to be analytic throughout a disk $|z| < R_0$ and series (1) becomes a *Maclaurin series*:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n \quad (|z| < R_0). \tag{4}$$

The proof when z_0 is arbitrary will follow as an immediate consequence. A reader who wishes to accept the proof of Taylor's theorem can easily skip to the examples in Sec. 59.

58. PROOF OF TAYLOR'S THEOREM

To begin the derivation of representation (4), Sec. 57, we write $|z| = r$ and let C_0 denote and positively oriented circle $|z| = r_0$, where $r < r_0 < R_0$ (see Fig. 75). Since f is analytic inside and on the circle C_0 and since the point z is interior to

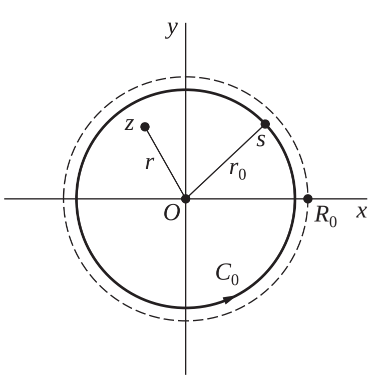

FIGURE 75

C_0, the Cauchy integral formula

$$f(z) = \frac{1}{2\pi i}\int_{C_0} \frac{f(s)\,ds}{s-z} \tag{1}$$

applies.

Now the factor $1/(s-z)$ in the integrand here can be put in the form

$$\frac{1}{s-z} = \frac{1}{s}\cdot\frac{1}{1-(z/s)}; \tag{2}$$

and we know from the example in Sec. 56 that

$$\frac{1}{1-z} = \sum_{n=0}^{N-1} z^n + \frac{z^N}{1-z} \tag{3}$$

when z is any complex number other than unity. Replacing z by z/s in expression (3), then, we can rewrite equation (2) as

$$\frac{1}{s-z} = \sum_{n=0}^{N-1} \frac{1}{s^{n+1}}\,z^n + z^N \frac{1}{(s-z)s^N}. \tag{4}$$

Multiplying through this equation by $f(s)$ and then integrating each side with respect to s around C_0, we find that

$$\int_{C_0} \frac{f(s)\;ds}{s-z} = \sum_{n=0}^{N-1} \int_{C_0} \frac{f(s)\;ds}{s^{n+1}}\,z^n + z^N \int_{C_0} \frac{f(s)\;ds}{(s-z)s^N}.$$

In view of expression (1) and the fact that (Sec. 51)

$$\frac{1}{2\pi i}\int_{C_0} \frac{f(s)\;ds}{s^{n+1}} = \frac{f^{(n)}(0)}{n!} \qquad (n = 0, 1, 2, \ldots),$$

this reduces, after we multiply through by $1/(2\pi i)$, to

$$f(z) = \sum_{n=0}^{N-1} \frac{f^{(n)}(0)}{n!}\,z^n + \rho_N(z), \tag{5}$$

where

$$\rho_N(z) = \frac{z^N}{2\pi i}\int_{C_0} \frac{f(s)\;ds}{(s-z)s^N}. \tag{6}$$

Representation (4) in Sec. 57 now follows once it is shown that

$$\lim_{N\to\infty} \rho_N(z) = 0. \tag{7}$$

To accomplish this, we recall that $|z| = r$ and that C_0 has radius r_0, where $r_0 > r$. Then, if s is a point on C_0, we can see that

$$|s - z| \geq ||s| - |z|| = r_0 - r.$$

Consequently, if M denotes the maximum value of $|f(s)|$ on C_0,

$$|\rho_N(z)| \leq \frac{r^N}{2\pi} \cdot \frac{M}{(r_0 - r)r_0^N} 2\pi r_0 = \frac{Mr_0}{r_0 - r}\left(\frac{r}{r_0}\right)^N.$$

Inasmuch as $(r/r_0) < 1$, limit (7) clearly holds.

To verify the theorem when the disk of radius R_0 is centered at an arbitrary point z_0, we suppose that f is analytic when $|z - z_0| < R_0$ and note that the composite function $f(z + z_0)$ must be analytic when $|(z + z_0) - z_0| < R_0$. This last inequality is, of course, just $|z| < R_0$; and, if we write $g(z) = f(z + z_0)$, the analyticity of g in the disk $|z| < R_0$ ensures the existence of a Maclaurin series representation:

$$g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} z^n \qquad (|z| < R_0).$$

That is,

$$f(z + z_0) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} z^n \qquad (|z| < R_0).$$

After replacing z by $z - z_0$ in this equation and its condition of validity, we have the desired Taylor series expansion (1) in Sec. 57.

59. EXAMPLES

In Sec. 66, we shall see that if there are constants a_n $(n = 0, 1, 2, \ldots)$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for all points z interior to some circle centered at z_0, then the power series here must be *the* Taylor series for f about z_0, regardless of how those constants arise. This observation often allows us to find the coefficients a_n in Taylor series in more efficient ways than by appealing directly to the formula $a_n = f^{(n)}(z_0)/n!$ in Taylor's theorem.

In the following examples, we use the formula in Taylor's theorem to find the Maclaurin series expansions of some fairly simple functions, and we emphasize the use of those expansions in finding other representations. In our examples, we shall freely use expected properties of convergent series, such as those verified in Exercises 7 and 8, Sec. 56.

EXAMPLE 1. Since the function $f(z) = e^z$ is entire, it has a Maclaurin series representation which is valid for all z. Here $f^{(n)}(z) = e^z$ $(n = 0, 1, 2, \ldots)$;